Digital Signal Processing

Mr. Nitin Kumar
Assistant Professor
ECE Department

Z TRANSFORM AND DFT

Z-Transform

Fourier Transform of a discrete time signal:

$$X(e^{jw}) = \sum_{k=-\infty}^{\infty} x(k)e^{-jwk}$$

Given a sequence **x(n)**, its **z transform** is defined:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Where **z** is a **complex variable** **$z=e^{j\omega}$**

- The **z transform does not converge** for all sequences or for all values of z
- The set of values of z for which the **z transform converges** is called **region of convergence**
- The properties of the sequence **x(n)** determines the **region of convergence of X(z)**

Z TRANSFORM AND DFT
Z-Transform

Finite-Length Sequences : FIR filters $$X(z) = \sum_{n=n_1}^{n_2} x(n) z^{-n}$$

Convergence requires **:** $|\, x(n) \,| < \infty \qquad n_1 \le n \le n_2$

z may take all values **except :** $z = \infty$ if $n_1 < 0$ and $z = 0$ if $n_2 > 0$

Region of convergence : $0 < |\, z \,| < \infty$

Compute X(z) :

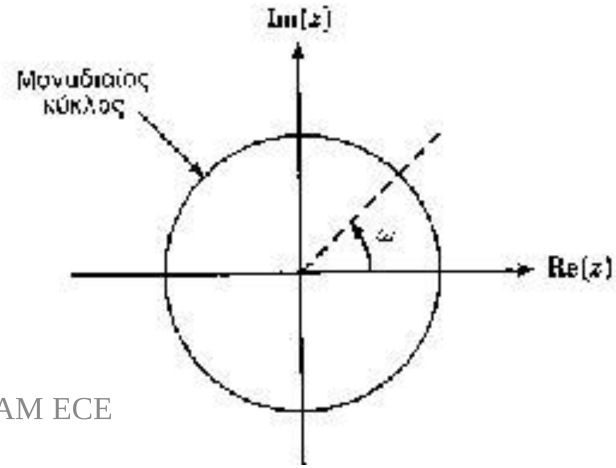
z = 1 (ω=0)

z = j (ω = π/2)

z = -1 (ω = π)

Unit circle inside

Region of convergence



Z TRANSFORM AND DFT
Z-Transform

In many cases **X(z)** is a **rational function :**

Ratio of polynomials

Values of **z** for which **X(z)=0** **Zeros** of **X(z)**

Values of **z** for which **X(z)=infinity** **Poles** of **X(z)**

- No **poles** of **X(z)** can occur within the **region of convergence** (is bounded by poles)
- Graphically display **z transform** by **pole-zero plot**

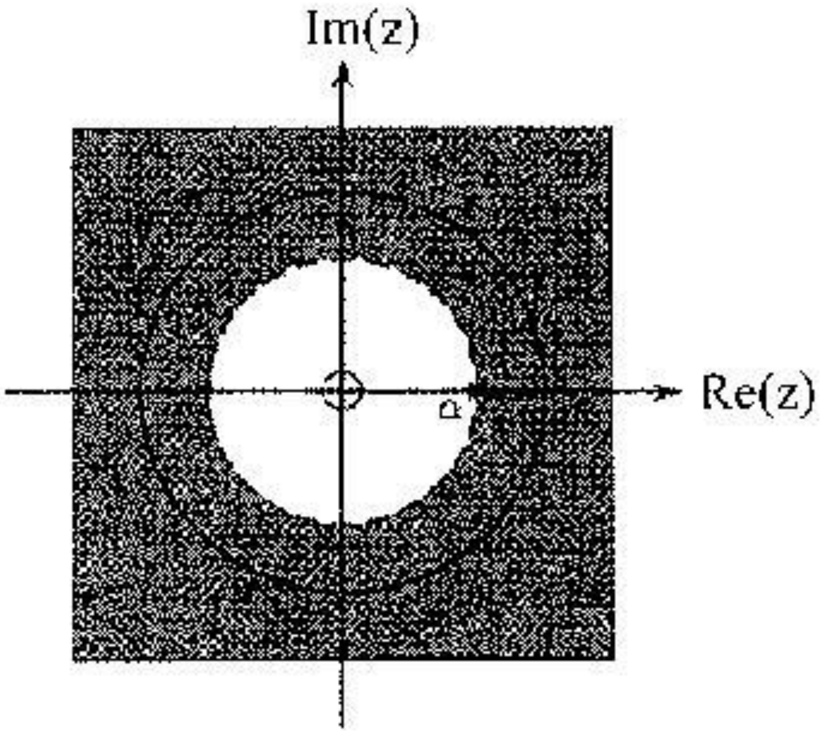
Example:

Compute the **z transform** of the sequence **$x(n)=a^n u(n)$**

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n =$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Z TRANSFORM AND DFT
Z-Transform



- If **|a|<1** the **unit circle** is included in the **region of convergence, X(z) converges**
- For **causal systems X(z) converges** everywhere outside a circle passing through the **pole farthest** from the origin of the z plane.

Z TRANSFORM AND DFT

Z-Transform

Ακολουθία	Μετασχηματισμός Z
$\delta(n)$	1
$\alpha^n u(n)$	$\dfrac{1}{1-\alpha z^{-1}}$
$-\alpha^n u(-n-1)$	$\dfrac{1}{1-\alpha z^{-1}}$
$n\alpha^n u(n)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$
$-n\alpha^n u(-n-1)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$
$\cos(n\omega_0)u(n)$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$
$\sin(n\omega_0)u(n)$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$

Z TRANSFORM AND DFT
Properties of the Z-Transform

1) **Linearity:** $ax_1(n) + bx_2(n) \Rightarrow aX_1(z) + bX_2(z)$

2) **Shifting:** $x(n-m) \Rightarrow z^{-m}X(z)$

3) **Time scaling by a Complex Exponential Sequence :** $a^n x(n) \Rightarrow X(a^{-1}z)$

4) **Convolution:** $y(n) = x(n) * h(n) \Rightarrow Y(z) = X(z)H(z)$

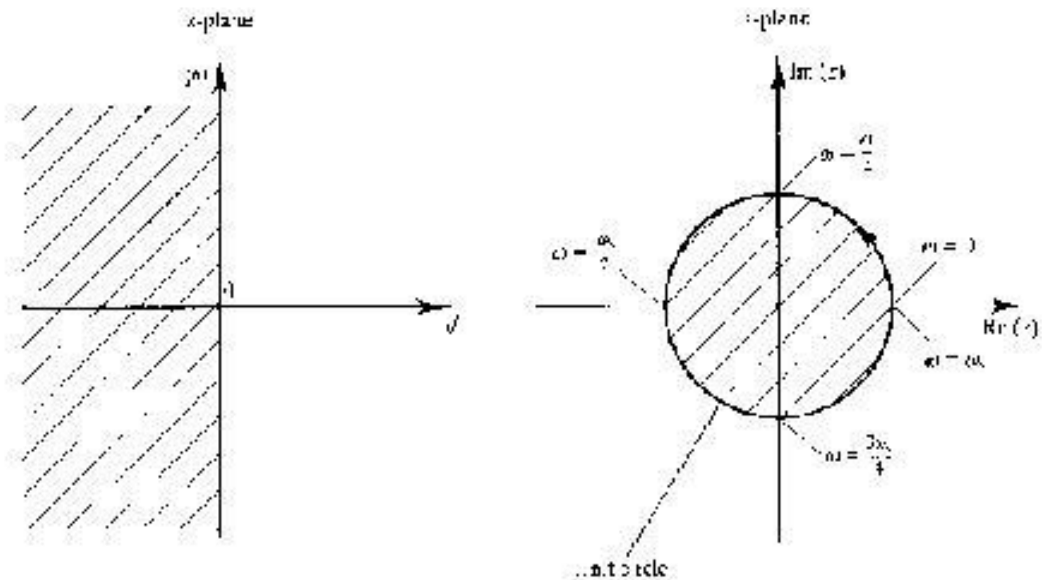
5) **Differentiation:** $nx(n) \Rightarrow -z\dfrac{dX(z)}{dz}$

Z TRANSFORM AND DFT

Relationship between Z-Transform and Laplace

- If **$z=e^{sT}$, $s=d+j\omega$**
  $z=e^{(d+j\omega)T}= e^{dT}\, e^{j\omega T}$
- Then,

$$|\, z \,| = e^{dT} \qquad \angle z = \omega T = 2\pi f / F_s = 2\pi\omega / \omega_s$$



- **Stability: Poles** should be inside the unit circle
- **Stability** criterion: Finding the poles of the system
- **FIR digital filters** always **stable**: Poles in origin

Z TRANSFORM AND DFT

Geometric Evaluation of Fourier Transform

- **X(z)** has **M zeros** at **$z=z_1,z_2,\ldots,z_M$**
- **X(z)** has **N poles** at **$z=p_1,p_2,\ldots,p_N$**
- We can write **X(z) in factored form:**

$$X(z) = A\frac{\prod_{i=1}^{M}(1 - z_i z^{-1})}{\prod_{i=1}^{N}(1 - p_i z^{-1})}$$

- Multiplying factors **X(z) can be written** as a **rational fraction:**

$$X(z) = \frac{\sum_{i=0}^{M} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}}$$

- This form is often used for **general filter design**

Z TRANSFORM AND DFT

Geometric Evaluation of Fourier Transform

The **Fourier transform** or **system function :**

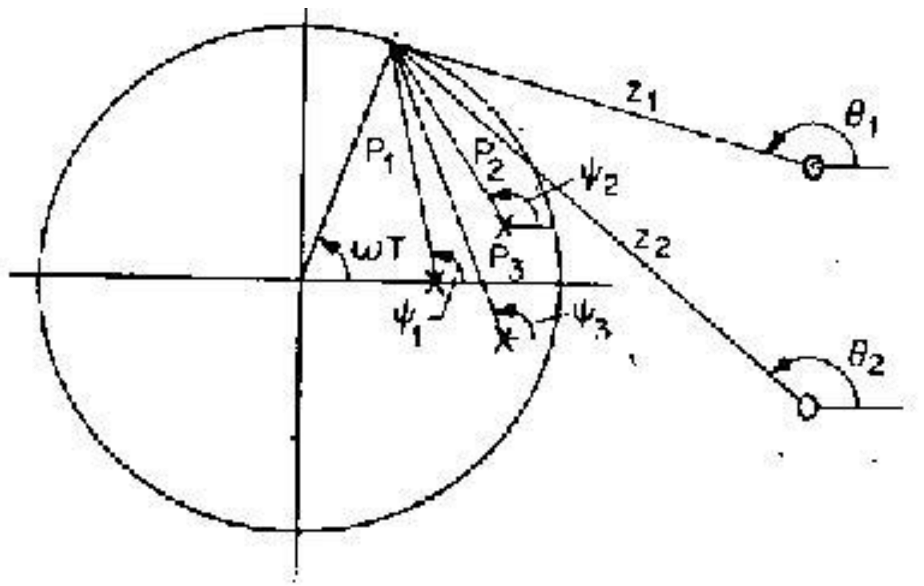
- Evaluating **X(z)** on the unit circle**,** **$z=e^{j\omega}$**

$$X(e^{j\omega}) = A\frac{\prod_{i=1}^{M}(1-z_i e^{-j\omega})}{\prod_{i=1}^{N}(1-p_i e^{-j\omega})}$$

$$|X(e^{j\omega})| = |A|\frac{\prod_{i=1}^{M}|1-z_i e^{-j\omega}|}{\prod_{i=1}^{N}|1-p_i e^{-j\omega}|} = |A|\frac{\prod_{i=1}^{M}|e^{j\omega}-z_i|}{\prod_{i=1}^{N}|e^{j\omega}-p_i|}$$

$$\angle X(e^{j\omega}) = \angle A + \sum_{i=1}^{M}\angle(1-z_i e^{-j\omega}) - \sum_{i=1}^{N}\angle(1-p_i e^{-j\omega})$$

Z TRANSFORM AND DFT
Geometric Evaluation of Fourier Transform



- From the point **z= e$^{j\omega}$** draw **vectors** to **zeros** and **poles**
- **Magnitudes** of vectors determine **magnitude** at **ω**
- **Angles** determine **phase**

Example :

$$|X(e^{j\omega})| = \frac{Z_1 Z_2}{P_1 P_2 P_3}$$

$$\angle X(e^{j\omega}) = \theta_1 + \theta_2 - (\psi_1 + \psi_2 + \psi_3)$$

Z TRANSFORM AND DFT

Inverse Z-Transform

From the **inverse z transform** we get **x(n)**

- Power series (long division)
- Partial fraction expansion
- Residue Theorem

Power series (long division)

- **X(z)** can be written as **rational fraction:**

$$X(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}$$

- It can be extended into an **infinite series** in **z^{-1}** by long division :

$$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + \ldots$$

Inverse Z-Transform

Example:

Find the **first 4** values of the sequence **f(k)**

$$F(z) = \frac{2z^{-1}}{2z^{-2} - 3z^{-1} + 1} = \frac{2z}{z^2 - 3z + 2}$$

$$\frac{2z}{z^2 - 3z + 2} = 2z^{-1} + 6z^{-2} + 14z^{-3} + \cdots$$

f(k)={0,2,6,14….}

- The long division approach can be reformulated so **x(n)** can be obtained recursively:

$$x(n) = \frac{\left[a_n - \sum_{i=1}^{n} x(n-i)b_i\right]}{b_0} \qquad n = 1, 2, ...$$

$$x(0) = \frac{a_0}{b_0}$$

Z TRANSFORM AND DFT
Inverse Z-Transform

Partial fraction expansion :

$$X(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}$$

- If poles of **X(z) first order (distinct)** and **N=M,**

$$\begin{aligned} X(z) &= B_0 + \frac{C_1}{1 - p_1 z^{-1}} + \frac{C_2}{1 - p_2 z^{-1}} + \ldots + \frac{C_M}{1 - p_M z^{-1}} \\ &= B_0 + \frac{C_1 z}{z - p_1} + \frac{C_2 z}{z - p_2} + \ldots + \frac{C_M z}{z - p_M} \end{aligned}$$

- **p(k): distinct poles, C_k partial fraction coef.**
- **$B_0 = a_0/b_0$**

- If **N<M** then **$B_0 = 0$**
- If **N>M** then by long division make **N<=M**

Z TRANSFORM AND DFT

Inverse Z-Transform

- The coefficient **C_k** can be derived as:

$$C_k = \frac{X(z)}{z}(z - p_k)\Big|_{z=p_k}$$

- If **X(z)** contains **multiple poles** extra terms are required - **X(z)** contains **m**th-order poles:

$$\sum_{i=1}^{m} \frac{D_i}{(z - p_k)^i}$$

$$D_i = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\Big[(z - p_k)^m X(z)\Big]\Big|_{z=p_k}$$

Z TRANSFORM AND DFT
Inverse Z-Transform

Example:

Find the **inverse z**-transform :

$$X(z) = \frac{2z^{-1}}{2z^{-2} - 3z^{-1} + 1} = \frac{2z}{z^2 - 3z + 2} = \frac{2z}{(z-2)(z-1)}$$

$$\frac{X(z)}{z} = \frac{2}{(z-2)(z-1)} = \frac{C_1}{(z-1)} + \frac{C_2}{(z-2)}$$

$$C_1 = \frac{X(z)}{z}(z-1)\Big|_{z=1} = \frac{2}{(1-2)} = -2$$

$$C_2 = \frac{X(z)}{z}(z-2)\Big|_{z=2} = \frac{2}{(2-1)} = 2$$

$$Z^{-1} = \left[\frac{z}{(z-a)}\right] = a^n, \qquad n \geq 0$$

$$Z^{-1} = \left[\frac{z}{(z-1)}\right] = u(n), \qquad n \geq 0$$

$$x(n) = 2 \cdot 2^n - 2, \qquad n \geq 0$$

Z TRANSFORM AND DFT

Inverse Z-Transform

Residue Theorem

IZT obtained by evaluating the contour integral:

$$x(n) = \frac{1}{2\pi j} \oint_C z^{n-1} X(z) dz$$

- Where **C** is the path of integration enclosing all the poles of X(z).

Cauchy's residue theorem:

- Sum of the residues of $z^{n-1}X(z)$ at all the poles inside **C**
- Every residue C_k, is associated with a pole at p_k

$$Res\left[z^{n-1}X(z), p_k\right] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-p_k) z^{n-1} X(z)\right]\Big|_{z=p_k}$$

- **m** is the order of the pole at $z=p_k$
- For a **first-order** pole:

$$Res\left[z^{n-1}X(z), p_k\right] = (z-p_k) z^{n-1} X(z) \Big|_{z=p_k}$$

Z TRANSFORM AND DFT
Inverse Z-Transform

Example:

Find the **inverse z** transform :

$$X(z) = \frac{z^2}{(z-0.5)(z-1)^2}$$

$$z^{n-1}X(z) = \frac{z^{n+1}}{(z-0.5)(z-1)^2}$$

Single pole @ **z=0.5, second-order** pole @ **z=1**

$$Res\left[z^{n-1}X(z), 0.5\right] = \frac{(z-0.5)z^{n+1}}{(z-0.5)(z-1)^2} = \frac{z^{n+1}}{(z-1)^2}\bigg|_{z=0.5}$$

$$= \frac{(0.5)^{n+1}}{(-0.5)^2} = 2(0.5)^n$$

$$Res\left[z^{n-1}X(z), 1\right] = \frac{d}{dz}\left[\frac{(z-1)^2 z^{n+1}}{(z-0.5)(z-1)^2}\right]$$

$$= \frac{(z-0.5)(n+1)z^n - z^{n+1}}{(z-0.5)^2}\bigg|_{z=1}$$

$$= \frac{(0.5)(n+1)-1}{(0.5)^2} = 2(n-1)$$

Z TRANSFORM AND DFT

Inverse Z-Transform

Combining the results we have:

$$x(n)=2[(n-1)+(0.5)^n]$$

No need to use inverse tables!!!

Comparison of the inverse z-transform

- **Power series:**

  Does not lead to a closed form solution, it is **simple**, easy computer implementation
- **Partial fraction, residue:**
  - **Closed form solution**,
  - Need to factorize polynomial (find poles of X(z))
  - May involve high order differentiation (multiple poles)
- **Partial fraction** : Useful in generating the coefficients of parallel structures for digital filters.
- **Residue method** : widely used in the analysis of quantization errors in discrete-time systems.

Z TRANSFORM AND DFT
Solving Difference Equations Using Z-Transform

The difference equation of interest (**IIR filters**) is:

$$y(n) = \sum_{i=0}^{N} b_i x(n-i) - \sum_{i=0}^{M} a_i y(n-i) \qquad n \geq 0$$

The z-transform is:

$$Y(z) = \sum_{i=0}^{N} b_i z^{-i} X(z) - \sum_{i=0}^{M} a_i z^{-i} Y(z)$$

Transfer function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} b_i z^{-i}}{1 + \sum_{i=0}^{M} a_i z^{-i}}$$

If coefficients **a_i=0** (**FIR filter**):

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{N} b_i z^{-i}$$

Z TRANSFORM AND DFT

Solving Difference Equations Using Z-Transform

Example:

Find the output of the following filter **: y(n) = x(n) + a y(n-1)**

Initial condition: **y(-1) = 0**

Input**: x(n) =** $e^{j\omega n}$ **u(n)**

Using z transform:

Y(z) = X(z) + a z^{-1} **Y(z)**

$$x(n) = e^{jwn}, \qquad X(z) = \frac{1}{1 - e^{jw} z^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

$$Y(z) = H(z)X(z) = \frac{1}{(1 - az^{-1})(1 - e^{jw} z^{-1})}$$

Using partial fraction expansion:

$$Y(z) = \frac{a/(a - e^{jw})}{(1 - az^{-1})} + \frac{-e^{jw}/(a - e^{jw})}{(1 - e^{jw} z^{-1})}$$

$$y(n) = \left[\frac{a^n}{a - e^{jw}} - \frac{e^{jwn}}{a - e^{jw}}\right] u(n)$$

Z TRANSFORM AND DFT

Discrete Fourier Transform (DFT)

- **Techniques for representing sequences:**
  - ❖ **Fourier Transform**
  - ❖ **Z-transform**
  - ❖ **Convolution summation**
- **Three** good reasons to study DFT
  - ❖ **It can be efficiently computed**
  - ❖ **Large number of applications**
    - Filter design
    - Fast convolution for FIR filtering
    - Approximation of other transforms
  - ❖ **Can be finitely parametrized**
- When a sequence is **periodic** or **of finite duration**, the sequence can be represented in a **discrete-Fourier series**
- **Periodic sequence x(n),** period **N,**

$$x(n) = \sum_{k=-\infty}^{\infty} X(k) e^{j(2\pi/N)kn}$$

Z TRANSFORM AND DFT

Discrete Fourier Transform (DFT)

- **Remember: $e^{j\omega}$** periodic with frequency **2π**
- **2π k n / N = 2π n ➔ k = N**
- **N** distinct exponentials

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn}$$

,1/N just a scale factor

- The **DFT** is defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = X(e^{jw})\Big|_{w=2\pi k/N}$$

- **DFT coefficients** correspond to **N** samples of **X(z) :**

$$X(k) = X(z)\Big|_{z=e^{j2\pi k/N}}$$

Z TRANSFORM AND DFT

Properties of the DFT

Linearity

If **x(n)** and **y(n)** are sequences **(N samples)** then:

a x(n) + b y(n) ⇔ a X(k) + b Y(k)

Remember: x(n) and **y(n)** must be **N samples,**

otherwise **zerofill**

Symmetry

If **x(n)** is a **real** sequence of **N samples** then:

Re[X(k)] = Re [X(N-k)]

Im[X(k)] = -Im[X(N-k)]

|X(K)| = |X(N-k)|

Phase X(k) = - Phase X(N-k)

If **x(n)** is **real** and **symmetric x(n) = x(N-n)** then:

X(K) is **purely real**

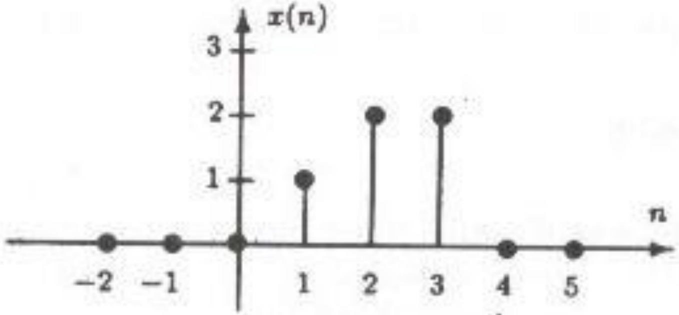
Z TRANSFORM AND DFT

Properties of the DFT

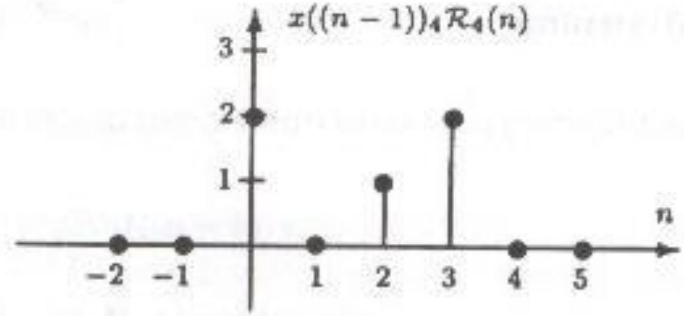
Shifting Property

If **x(n)** is **periodic** then **x(n) ←→ X(k),** $x(n-n_0) \leftrightarrow X(k)\, e^{-j(2\pi/N)\, n_0 k}$

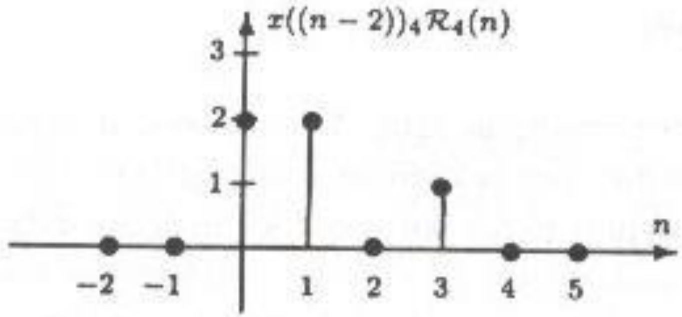
If **x(n)** is **not periodic** then **time-shift** is created by **rotating x(n) circularly** by n_0 samples.



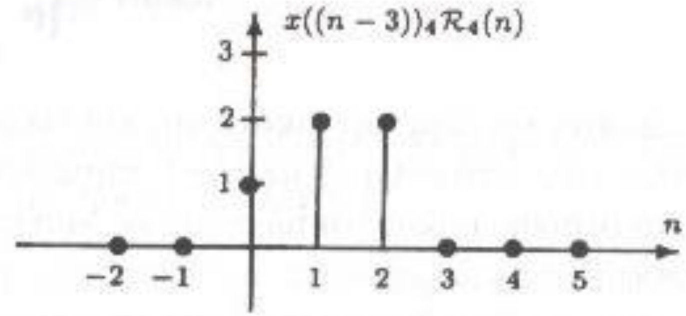
(α) Ένα σήμα διακριτού χρόνου μήκους N=4.



(β) Κυκλική μετατόπιση κατά ένα.



(γ) Κυκλική μετατόπιση κατά δύο.



(δ) Κυκλική μετατόπιση κατά τρία.

Z TRANSFORM AND DFT

Convolution of Sequences

If **x(n), h(n)** are **periodic sequences** period **N, DFTs:**

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-j(2\pi/N)nk}$$

y(n): circular convolution of x(n), h(n)

Y(k) N-point **DFT** of **y(n)**

$$y(n) = \sum_{l=0}^{N-1} x(l) h(n-l) = x(n) \otimes h(n)$$

$$Y(k) = X(k) \cdot H(k)$$

Linear convolution has **infinite sum.**

$$y(n) = \sum_{l=-\infty}^{\infty} x(l) h(n-l)$$

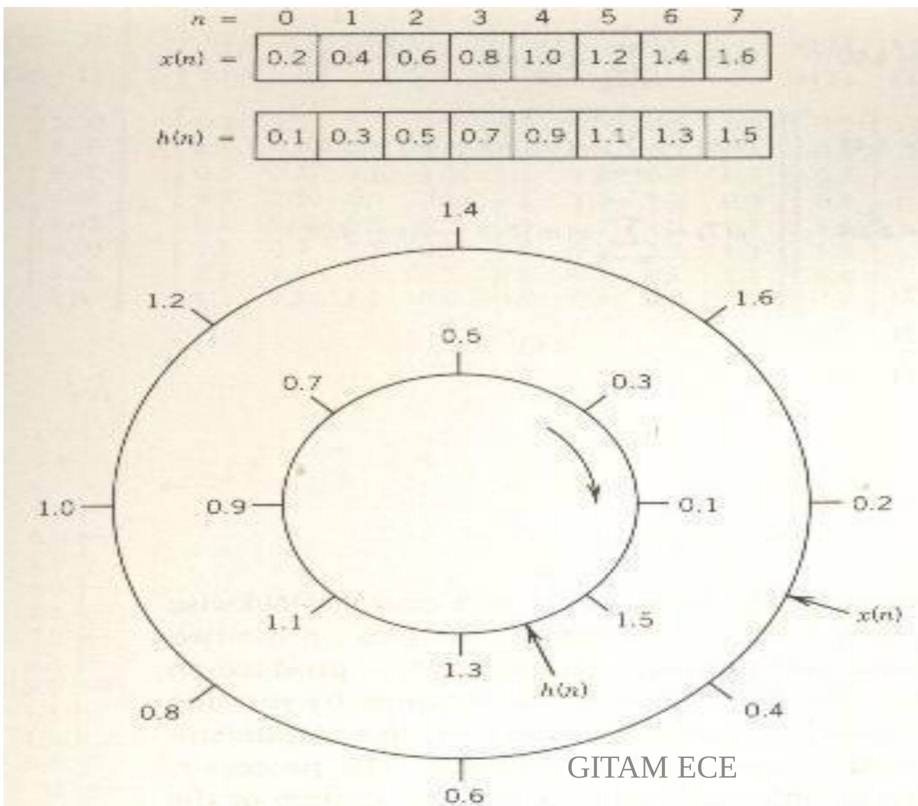
Z TRANSFORM AND DFT
Convolution of Sequences

- Imagine **one** sequence around a circle **N** points.
- **Second** sequence around a circle **N** points but **timed reversed**

Convolution: multiply values of **2** circles, **shift**

multiply, shift, …… N times

Example:



Z TRANSFORM AND DFT

Convolution of Sequences

$n =$	−1	−2	−3	−4	−5	−6	−7	0	1	2	3	4	5	6	7	…
$x(n)$	0.4	0.6	0.8	1.0	1.2	1.4	1.6	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	…
$h(-m)$	1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1		Rotate inner circle						$y(0)$
$h(1-m)$		1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	one sample clockwise						$y(1)$
$h(2-m)$			1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1		″				$y(2)$
$h(3-m)$				1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	″				$y(3)$
$h(4-m)$					1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	″			$y(4)$
$h(5-m)$						1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	″		$y(5)$
$h(6-m)$							1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	″	$y(6)$
$h(7-m)$								1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	$y(7)$

Process repeats with period $N = 8$

$$y(n) = \sum_{m=0}^{N-1} x(m)h(n-m)$$

$$y(0) = \sum_{m=0}^{7} x(m)h(0-m) = 5.20 \qquad y(4) = \sum_{m=0}^{7} x(m)h(4-m) = 6.48$$

$$y(1) = \sum_{m=0}^{7} x(m)b(1-m) = 6.00 \qquad y(5) = \sum_{m=0}^{7} x(m)h(5-m) = 6.00$$

$$y(2) = \sum_{m=0}^{7} x(m)h(2-m) = 6.48 \qquad y(6) = \sum_{m=0}^{7} x(m)h(6-m) = 5.20$$

$$y(3) = \sum_{m=0}^{7} x(m)h(3-m) = 6.64 \qquad y(7) = \sum_{m=0}^{7} x(m)h(7-m) = 4.08$$

$h(-1) = h(7)$ $\quad h(-5) = h(3)$

$h(-2) = h(6)$ $\quad h(-6) = h(2)$

$h(-3) = h(5)$ $\quad h(-7) = h(1)$

$h(-4) = h(4)$

Z TRANSFORM AND DFT
Sectioned Convolution

- **Fast Convolution:** Using DFT for 2 finite sequences
  - ❖ Evaluated **Rapidly**, **efficiently** with **FFT**
  - ❖ $N_1+N_2 > 30$ → **Fast Convolution** more **efficient**
- **Direct Convolution:** direct evaluation
- $L > N_1+N_2$ Add **zeros** to achieve **L power of 2**

Sectioned Convolution

- $N_1 >> N_2$,what to do?
- $L > N_1+N_2$ **,inefficient** and **impractical. Why?**
- **Long sequence** must be available **before convolution**
  - ❖ **Practical waveforms : Speech, Radar -** not available
  - ❖ **No processing** before entire sequence - **Long delays**
- **Solution: Sectioned Convolution**
- **Overlap – Add**
- **Overlap – Save**

Z TRANSFORM AND DFT
Sectioned Convolution

Overlap – Add

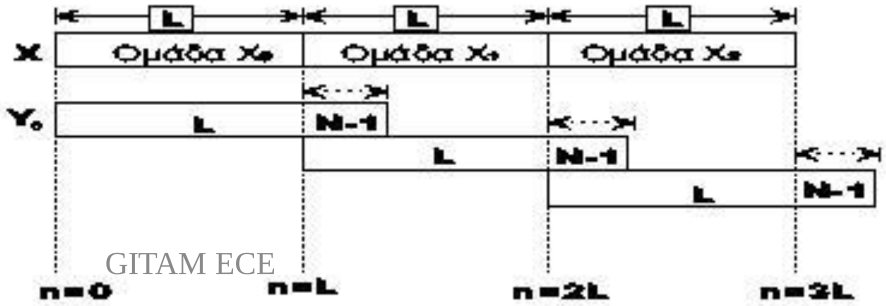
- Long sequence **x(n) infinite** duration
- Short sequence **h(n) N_2** duration
- **x(n)** is sectioned **N_3** or **L** or **M**

$$x(n) = \sum_{k=0}^{\infty} x_k(n)$$

$$x_k(n) = \begin{cases} x(n) & kN_3 \le n \le (k+1)N_3 - 1, \\ 0 & \text{αλλού.} \end{cases}$$

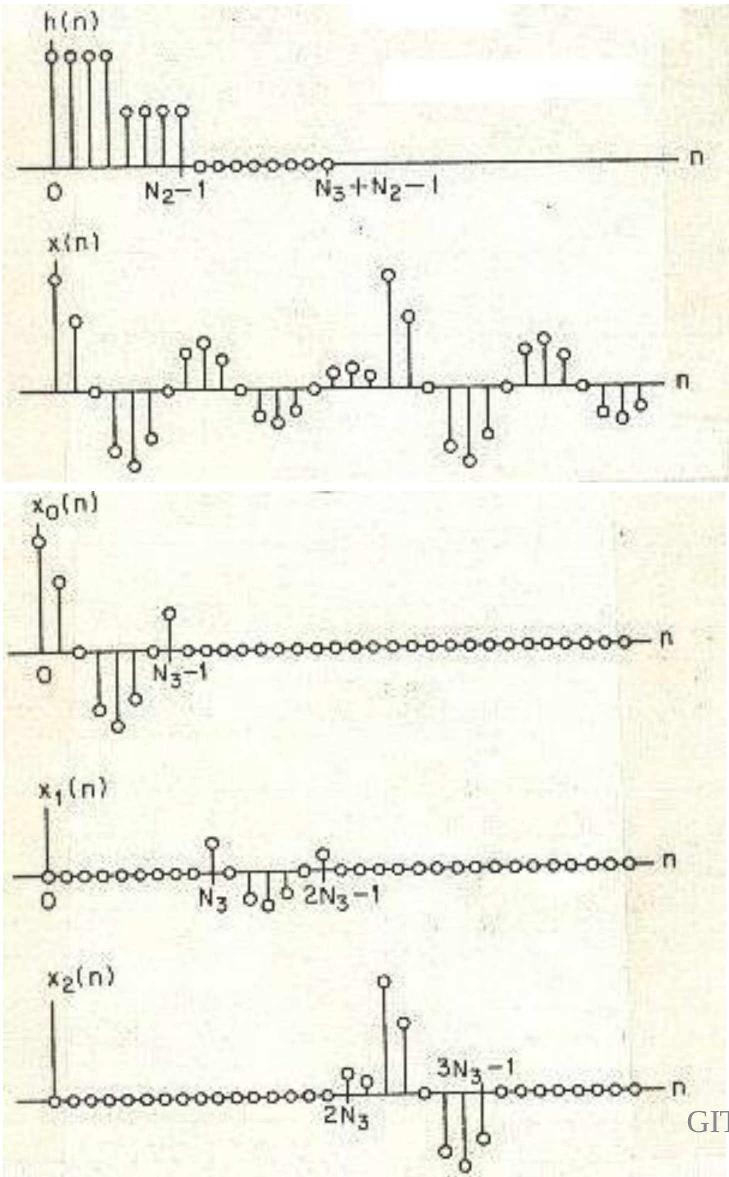
$$y(n) = \sum_{m=0}^{n} h(m) \sum_{k=0}^{\infty} x_k(n-m) = \sum_{k=0}^{\infty} y_k(n)$$

- Duration of **each** convolution **$N_3 + N_2 - 1$** (overlap)



Z TRANSFORM AND DFT

Sectioned Convolution



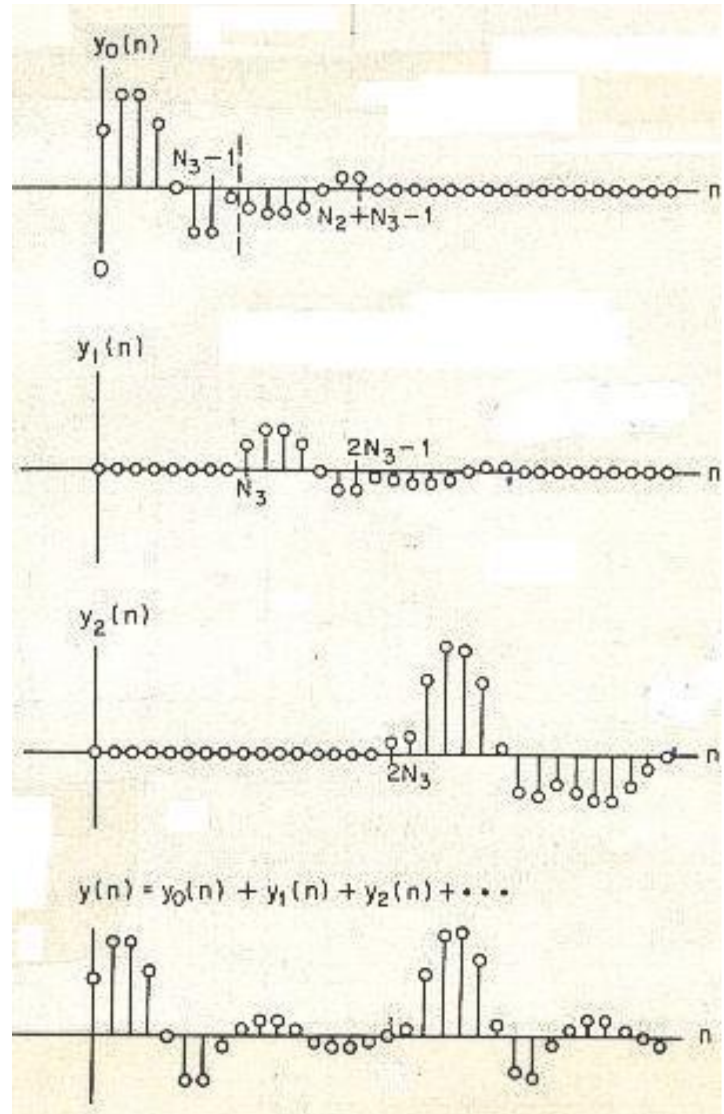


Fig. 2.33. Outputs of overlap-add method.

Thank You