Computer graphics

Two Dimensional Geometric Transformations

Translation

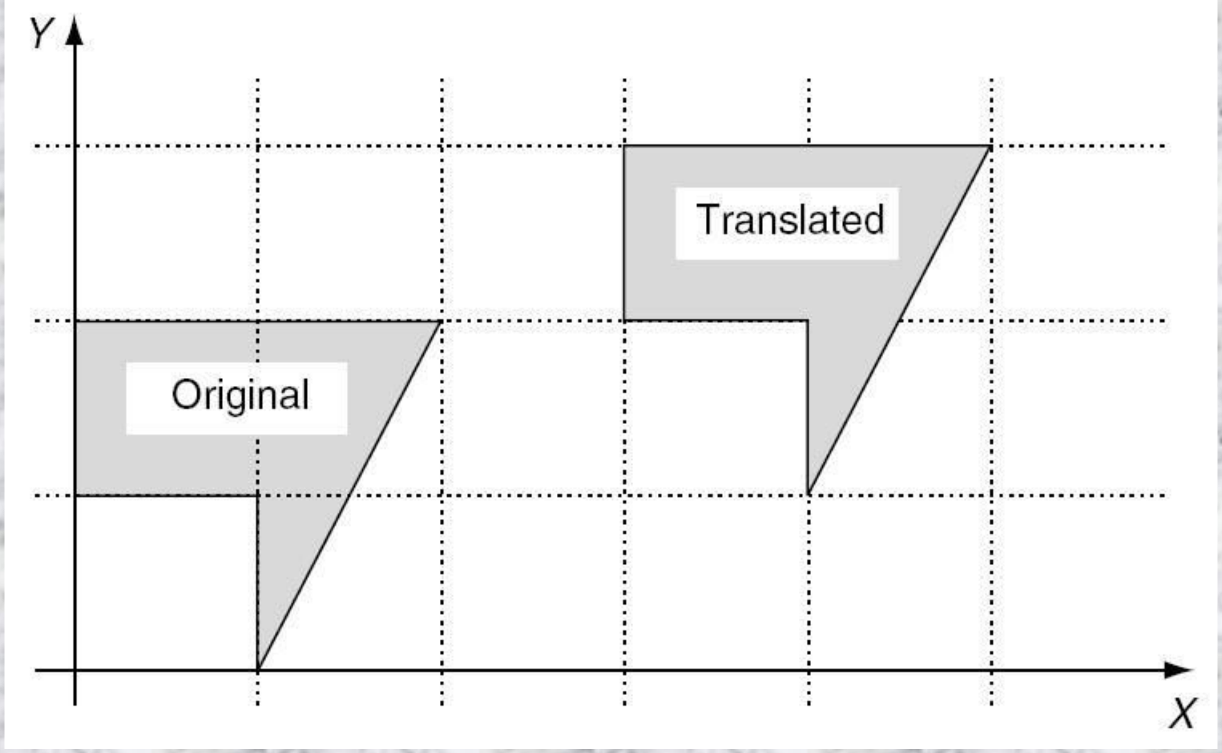
Cartesian coordinates provide a one-to-one relationship between number and shape, such that when we change a shape's coordinates, we change its geometry.

For example, if P(x, y) is a vertex on a shape, when we apply the operation $x'=x+3$ we create a new point P(x', y) three units to the right.

Similarly, the operation $y'=y+1$ creates a new point P(x, y') displaced one unit vertically.

By applying both of these transforms to every vertex to the original shape, the shape is displaced as shown in Figure.

Translation



The translated shape results by adding 3 to every x-coordinate, and 1 to every y-coordinate of the original shape.

Translation

The algebraic and matrix notation for 2D translation is

$$x' = x + t_x$$
$$y' = y + t_y$$

or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

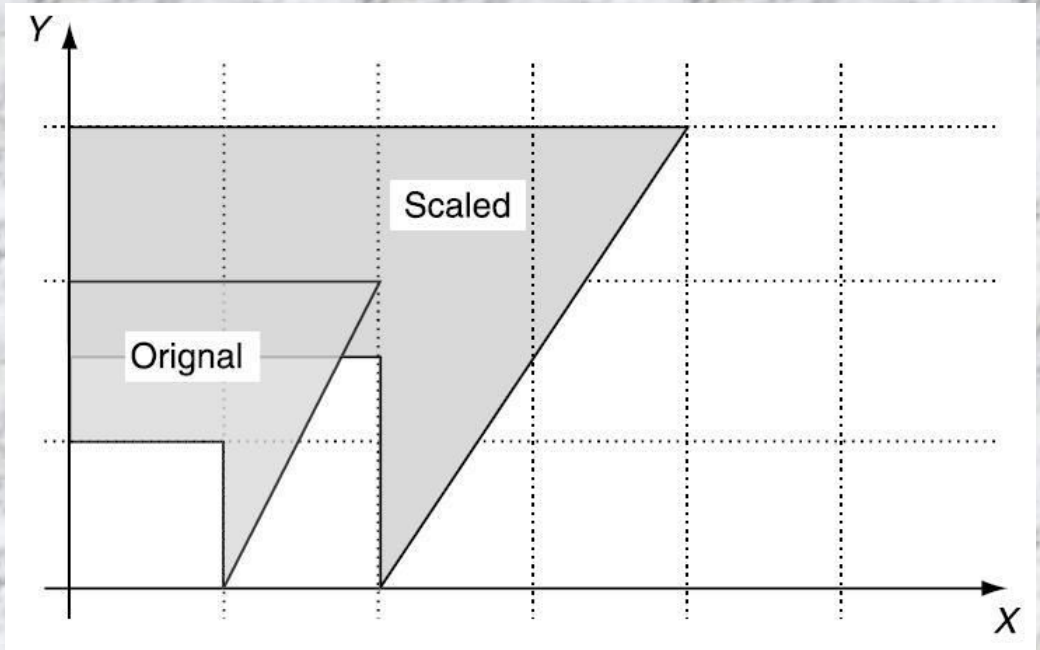
Scaling

Shape scaling is achieved by multiplying coordinates as follows:

$$x' = 2x \quad \text{and} \quad y' = 1.5y$$

This transform results in a horizontal scaling of **2** and a vertical scaling of **1.5**

Note that a point located at the origin does not change its place, so scaling is relative to the origin.



Scaling

The algebraic and matrix notation for 2D scaling is

$$\mathbf{x' = s_x\, x}$$

$$\mathbf{y' = s_y\, y}$$

or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The scaling action is relative to the **origin**, i.e. the point **(0,0)** remains (0,0). All other points move away from the origin.

Scaling

To scale relative to another point $(\mathbf{p_x,p_y})$ we **first** subtract (p_x,p_y) from (x,y) respectively. This effectively **translates** the reference point (p_x,p_y) back to the origin. **Second**, we perform the **scaling** operation, and **third**, add (p_x,p_y) back to (x,y) respectively, to compensate for the original subtraction. Algebraically this is

$$x' = s_x (x - p_x) + p_x$$

$$y' = s_y (y - p_y) + p_y$$

which simplifies to

$$x' = s_x x + p_x(1 - s_x)$$

$$y' = s_y y + p_y(1 - s_y)$$

Scaling

in a homogeneous matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & p_x(1-s_x) \\ 0 & s_y & p_y(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For example, to scale a shape by **2** relative to the point (**1, 1**) the matrix is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling

The strategy used to scale a point (x, y) relative to some arbitrary point **(p_x, p_y)** was to **first**, translate **($-p_x$, $-p_y$)**; **second**, perform the scaling; and **third**, translate **(p_x, p_y)**.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(px, py)]\,[\text{scale}(sx, sy)]\,[\text{translate}(-px, -py)] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection

To make a reflection of a shape relative to the **y-axis**, we simply reverse the sign of the x-coordinate, leaving the y-coordinate unchanged

$$\mathbf{x' = -x}$$

$$\mathbf{y' = y}$$

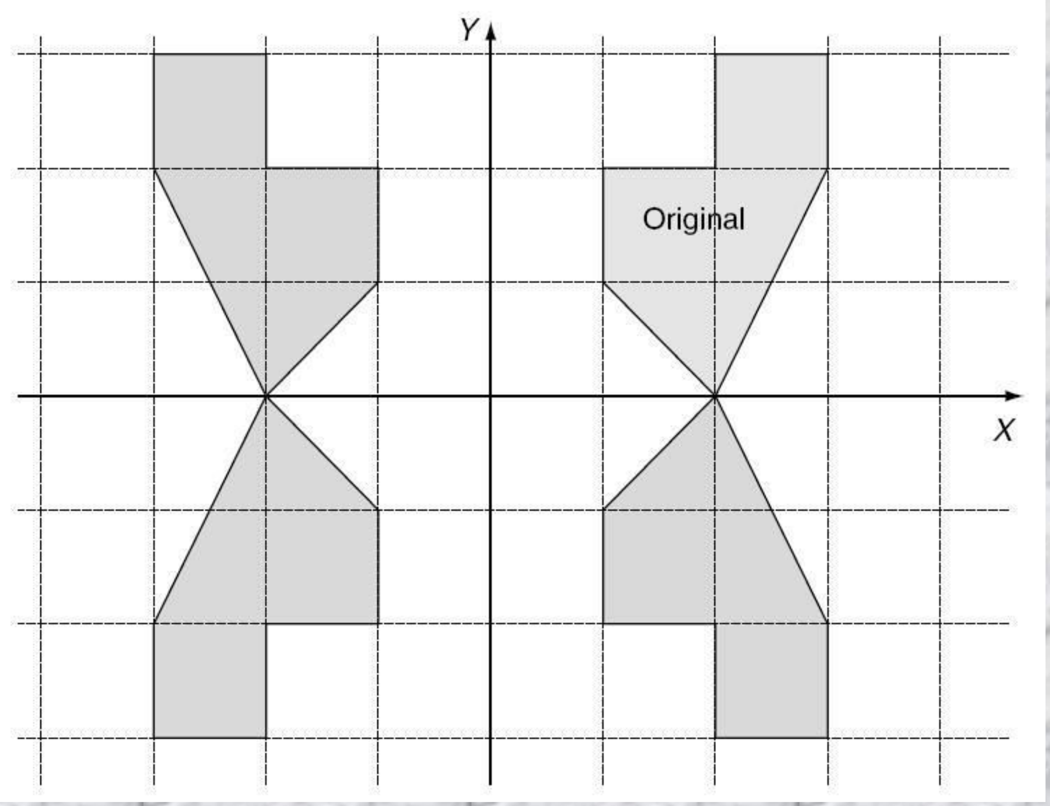
And to reflect a shape relative to the **x-axis** we reverse the y-coordinates:

$$\mathbf{x' = x}$$

$$\mathbf{y' = -y}$$

Reflection

Examples of reflections are shown in Figure. The top right-hand shape can give rise to the three reflections simply by reversing the signs of coordinates



Reflection

The matrix notation for reflecting about the **y-axis** is:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

about the **x-axis** is:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection

To make a reflection about an arbitrary vertical or horizontal axis we need to introduce some more algebraic deception. **For example**, to make a reflection about the **vertical axis x=1**

- First subtract 1 from the x -coordinate. This effectively makes the x=1 axis coincident with the major y-axis.

- Next we perform the reflection by reversing the sign of the modified x -coordinate.

- Finally, we add 1 to the reflected coordinate to compensate for the original subtraction. Algebraically, the three steps are

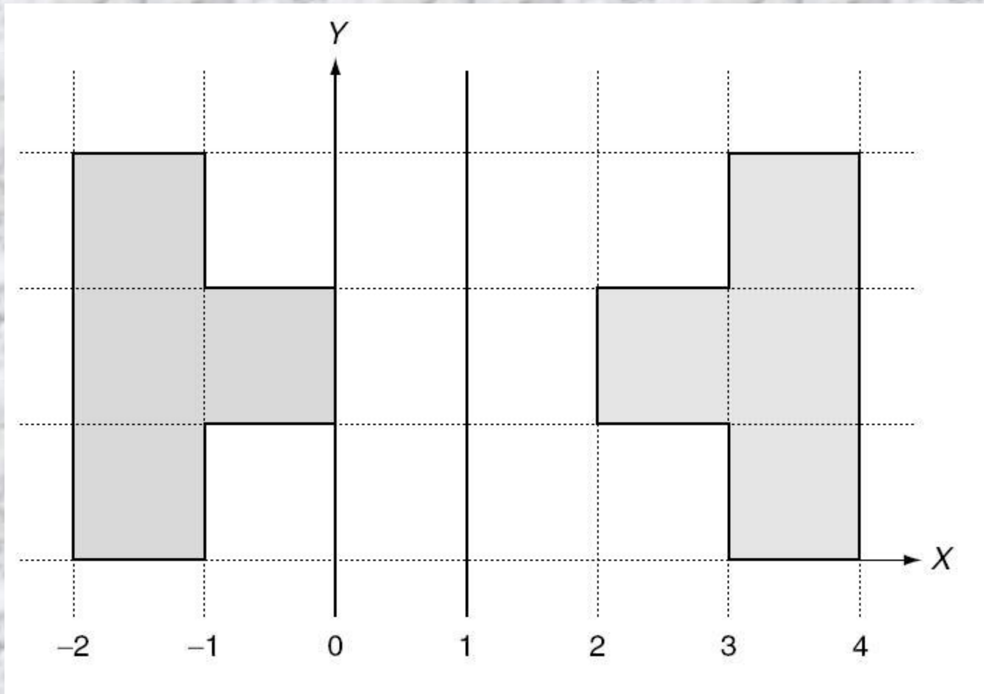
$$\mathbf{x_1 = x - 1 \qquad x_2 = -(x - 1) \qquad x' = -(x - 1) + 1}$$

which simplifies to $\mathbf{x' = -x + 2 \qquad y' = y}$

Reflection

In matrix form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



This figure illustrates this process.

The shape on the right is reflected about the x = 1 axis

Reflection

In general, to reflect a shape about an arbitrary **y-axis**, **$x = a_x$**, the following transform is required:

$$x' = -(x - a_x) + a_x = -x + 2a_x$$

$$y' = y$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection

Similarly, this transform is used for reflections about an arbitrary **x-axis**, $\mathbf{y = a_y}$:

$$x' = x$$

$$y' = -(y - a_y) + a_y = -y + 2a_y$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2a_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection

Therefore, using matrices, we can reason that a reflection transform about an arbitrary axis $\mathbf{x = a_x}$, parallel with the y-axis, is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \text{[translate(ax, 0)] [reflection] [translate(–ax,0)]} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

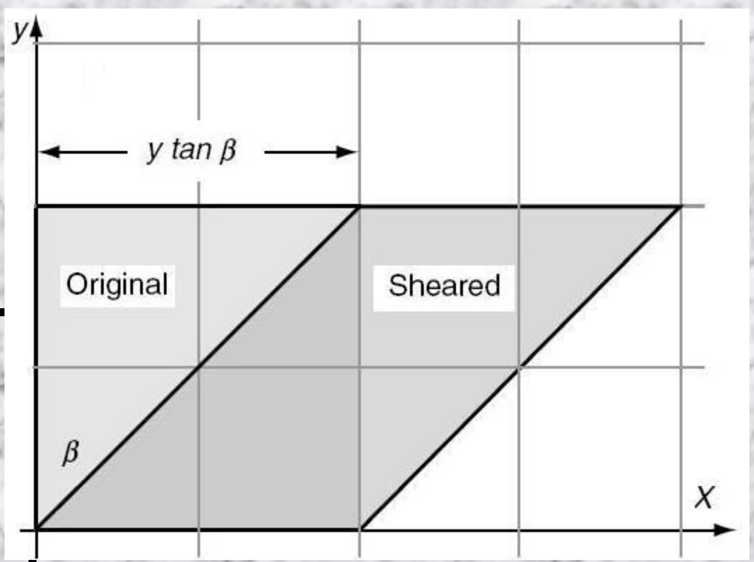
Shearing

A shape is sheared by leaning it over at an angle **β**. Figure below illustrates the geometry, and we see that the y-coordinate remains unchanged but the x -coordinate is a function of y and **tan(β).**

$$\mathbf{x' = x + y\tan(\beta)}$$

$$\mathbf{y' = y}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan(\beta) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



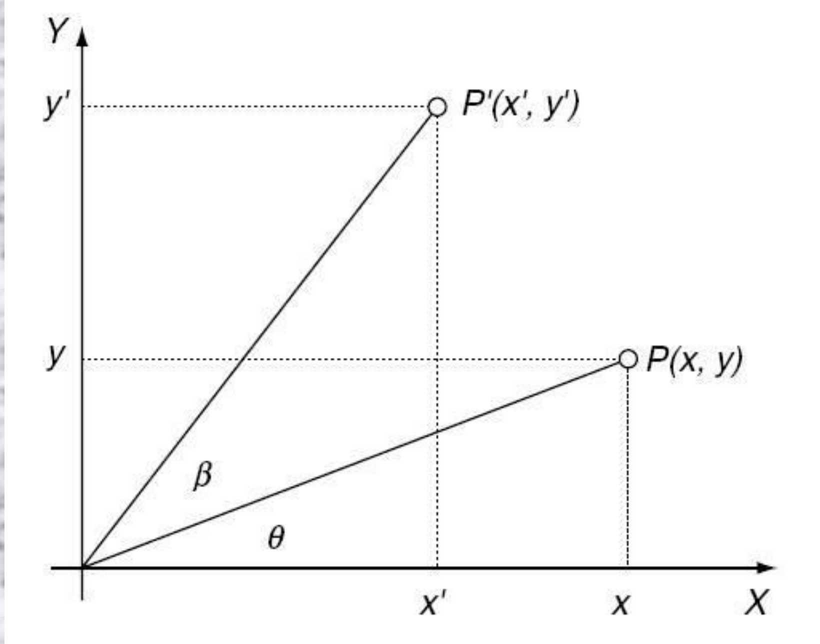
The original square shape is sheared to the right by an angle β, and the horizontal shift is proportional to y tan(β).

Rotation

In the Figure the point **P(x, y)** is to be rotated by an angle **β** about the **origin** to **P(x′, y′).** It can be seen that:

$$\mathbf{x' = R\cos(\theta + \beta)}$$

$$\mathbf{y' = R\sin(\theta + \beta)}$$



$$\mathbf{x'} = R(\cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta))$$

$$= R(\ (x/R)\cos(\beta) - (y/R)\sin(\beta)) \mathbf{= x\cos(\beta) - y\sin(\beta)}$$

$$\mathbf{y'} = R(\sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta))$$

$$= R(\ (y/R)\cos(\beta) + (x/R)\sin(\beta)) \mathbf{= x\sin(\beta) + y\cos(\beta)}$$

Rotation

$$x' = x\cos(\beta) - y\sin(\beta)$$

$$y' = x\sin(\beta) + y\cos(\beta)$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For example, to rotate a point by 90^0 the matrix becomes:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Thus the point (1, 0) becomes (0, 1).

Rotation

To rotate a point (x, y) about an arbitrary point (p_x, p_y)

1. Subtract (p_x, p_y) from the coordinates (x, y) respectively This enables us to perform the rotation about the origin.

$$x_1 = (x - p_x)$$
$$y_1 = (y - p_y)$$

2. Rotate β about the origin:

$$x_2 = (x - p_x) \cos(\beta) - (y - p_y) \sin(\beta)$$
$$y_2 = (x - p_x) \sin(\beta) + (y - p_y) \cos(\beta)$$

3. Add (p_x, p_y) to compensate for the original subtraction

$$x' = (x - p_x) \cos(\beta) - (y - p_y) \sin(\beta) + p_x$$
$$y' = (x - p_x) \sin(\beta) + (y - p_y) \cos(\beta) + p_y$$

Rotation

$$x' = x\cos(\beta) - y\sin(\beta) + p_x(1 - \cos(\beta)) + p_y\sin(\beta)$$
$$y' = x\sin(\beta) + y\cos(\beta) + p_y(1 - \cos(\beta)) - p_x\sin(\beta)$$

in matrix form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & p_x(1-\cos(\beta))+p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & p_y(1-\cos(\beta))-p_x\sin(\beta) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotating a point 90° about the point (1, 1) the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

$$x' = x\cos(\beta) - y\sin(\beta) + p_x(1 - \cos(\beta)) + p_y\sin(\beta)$$
$$y' = x\sin(\beta) + y\cos(\beta) + p_y(1 - \cos(\beta)) - p_x\sin(\beta)$$

in matrix form:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & p_x(1-\cos(\beta))+p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & p_y(1-\cos(\beta))-p_x\sin(\beta) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotating a point 90° about the point (1, 1) the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

Therefore, using matrices, we can develop a rotation about an arbitrary point (p_x, p_y) as follows:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = [\text{translate}(px, py)]\ [\text{rotate } \beta]\ [\text{translate}(-px, -py)] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -p_x \\ 0 & 1 & -p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$